Abstract

In this paper, we study the stationary node distribution of a variation of the Random Waypoint mobility model, in which nodes move in a smooth way following one randomly chosen Manhattan path connecting two points. We provide analytical results for the spatial node stationary distribution of this model. As an application, we exploit this result to derive an upper bound on the transmission range of the nodes of a MANET, moving according to this model, that guarantees the connectivity of the communication graph with high probability.

1 Introduction

The Random WayPoint (in short, RWP) mobility model is one of the most commonly used models for evaluating the performance of a communication protocol and/or application based on a mobile wireless ad hoc network (in short, MANET) [12]. According to this model, each node moves by selecting a random destination point $D$ within a specified movement region and a random velocity value $v$, and by traveling from its current position $S$ to $D$ at constant velocity $v$ along the segment joining $S$ to $D$ (for a survey on mobility models for MANET research, see [5]).

In [8] a variation of the RWP model, called random Manhattan RWP mobility model (in short, rMRWP), has been introduced and analyzed: according to this model, once the source and the destination points, and the velocity value have been chosen, the path followed by a node while moving from the source point to the destination point is one of the two Manhattan 2-segment paths connecting the two points, which is randomly chosen out of the two possible ones. By applying relatively simple geometric probability arguments similar to those used in [3, 13], the authors derived an exact closed formula for the spatial node stationary distribution of this model. Later, the same formula was also obtained in [6] by using a more general approach. This formula has then been used to compute an upper bound on the transmission range of the nodes of a MANET moving according to the rMRWP mobility model, that guarantees with high probability the connectivity of the communication graph, once the network has reached the stationary distribution. It is worth noting that the random Manhattan RWP has also been analyzed in [9] while considering the problem of load-balanced routing in dense wireless multi-hop networks. Moreover, in [7] the authors analyzed the completion time of the flooding protocol in MANET moving according to the rMRWP mobility model.

Passing through a right angle at constant speed might be considered unrealistic: for this reason, in this paper we introduce and analyze a variation of the rMRWP mobility model, in which the movement of the nodes is smooth. The advantage of introducing the acceleration within a mobility model has already been observed in [2], where the author experimentally shows how a smooth movement produces a more uniform spatial node distribution. In this paper, we propose the acceleration/deceleration random Manhattan RWP mobility model (in short, AC/DC-rMRWP) which differs from the rMRWP model since, once the speed value $v$ has been chosen, the node starts at zero velocity, travels according to a uniform accelerated motion until it reaches the mid-point of the first segment of the Manhattan path, travels according to a uniform decelerated motion until it reaches the turning point of the Manhattan path, and then repeats the process on the second segment of the Manhattan path.

Even in the case of the AC/DC-rMRWP mobility model, we will be able to derive an exact closed formula for the spatial node stationary distribution and, hence, to compute an upper bound on the transmission range required for guaranteeing the connectivity of the communication graph. Interestingly enough, the spatial node distribution of the AC/DC-rMRWP is closer to the uniform distribution than the spatial node distribution of the rMRWP (see Figure 1): as a consequence of this result, the transmission range turns out to be bounded by a smaller value. In a certain sense, we could say that considering a more realistic mobility model turns out to be also useful from a communication point of view.

The paper is organized as follows. In Section 2 we introduce the approach which will be used while deriving the spatial node distribution formula. Section 3 is devoted to the analytical derivation of this formula, while in Section 4 the upper bound on the communication range is proved. Finally, we conclude in Section 5.

Formal definitions 

Let $S = (x_S, y_S)$ and $D = (x_D, y_D)$ be two points in the square $Q = [0, 2] \times [0, 2]$ of center $Z = (1, 1)$. The Manhattan paths from $S$ to $D$ are $m_{hv}(S, D)$ and $m_{vh}(S, D)$, where $m_{hv}(S, D)$ is the horizontal path from $S$ to $H = (x_D, y_S)$ followed by the vertical path from $H$ to $D$, and $m_{vh}(S, D)$ is the vertical path from $S$ to $K = (x_S, y_D)$ followed by the horizontal path from $K$ to $D$.

The AC/DC-rMRWP mobility model is defined as follows.
Each node is initially positioned at a point \( S \), randomly chosen within \( Q \). Successively, the node chooses a random destination point \( D \in Q \), and a random velocity value \( v \in [v_{\text{m}},v_{\text{M}}] \) with \( v_{\text{m}} > 0 \). Then the node travels along a path randomly chosen between \( m_{\text{vh}}(S,D) \) and \( m_{\text{vh}}(S,D) \) according to the following AC/DC motion rule: if the chosen path is \( m_{\text{vh}}(S,D) \), then (1) the node travels at constant acceleration along the segment \( SH \) until it reaches its mid-point with velocity \( v \), (2) the node travels at constant deceleration along the second half of segment \( SH \) until it reaches \( H \) with velocity 0, (3) the node travels at constant acceleration along the segment \( HD \) until it reaches its mid-point at velocity \( v \), and (4) the node travels at constant deceleration along the second half of segment \( HD \) until it reaches \( D \) with velocity 0. The case in which the chosen path is \( m_{\text{vh}}(S,D) \) is defined similarly. Once the destination point \( D \) is reached, the node immediately starts the traveling process again.\(^2\)

2 The approach description

In this section, we describe the approach that will be followed while deriving the explicit formula for the spatial node stationary distribution of the AC/DC-rMRWP mobility models. Observe that, since nodes move independently, we can limit ourselves to analyze the movement of a single node.

For any \( p \) and \( q \) with \( 0 \leq p,q \leq 2 \), let \( R \) be the rectangle \([0,p] \times [0,q] \), and let \( F = (p,q) \) be the top right vertex of \( R \). Moreover, let \( X \) be the random variable describing the location of the node, and let \( T \) and \( T_{p,q} \) be the two random variables describing, respectively, the time spent by the node while moving between the source and the destination point and the time spent within \( R \) while moving between these two points. As proved in [3],

\[
\Pr(X \in R) = \frac{E[T_{p,q}]}{E[T]},
\]

\(^2\)In the RWP mobility model it is also assumed that, once a node reaches its destination, it stays there for a pause time \( t_p \), randomly chosen within a specified interval. In this paper, we assume that \( t_p = 0 \) — the case in which \( t_p > 0 \) can be dealt with similarly to what has been done in [4].

In other words, the problem of computing the cumulative distribution function (in short, cdf) \( \Pr(X \in R) \) has been reduced to the problem of computing the values \( E[T_{p,q}] \) and \( E[T] \).

Let \( S \) denote the location of a node at the beginning of its movement period and \( D \) denote the location of the same node at the end of the same movement period. Hence, the value of \( E[T_{p,q}] \) is equal to

\[
\frac{1}{\Delta v} \int_{v_m}^{v_M} \int_{S \in Q} \int_{D \in Q} f(S)f(D)t_R(S,D,v)dDdSdv,
\]

where \( f(.) \) denotes the probability density function\(^3\) (in short, pdf) of a point location, \( \frac{1}{\Delta v} = \frac{1}{v_M-v_m} \) is the pdf of the chosen velocity value, \( v \), and \( t_R(S,D,v) \) denotes the expected time spent within \( R \) while moving between \( S \) and \( D \) with velocity \( v \). The computation of the above integral will be performed by distinguishing the three cases in which (i) both \( S \) and \( D \) are contained in \( R \) or (ii) \( S \) is contained in \( R \) while \( D \) is outside of \( R \) (or vice versa) or (iii) both \( S \) and \( D \) are outside of \( R \). In particular, assuming \( S \) and \( D \) uniformly chosen in \( Q \), the value of the above integral can be obtained by computing the following four integrals:

\[
\Sigma_1 = \frac{1}{16\Delta v} \int_{v_m}^{v_M} \int_{S \in Q} \int_{D \in R} t_R(S,D,v)dDdSdv,
\]

\[
\Sigma_2 = \frac{1}{16\Delta v} \int_{v_m}^{v_M} \int_{S \in Q} \int_{D \in Q-R} t_R(S,D,v)dDdSdv,
\]

\[
\Sigma_3 = \frac{1}{16\Delta v} \int_{v_m}^{v_M} \int_{S \in Q-R} \int_{D \in Q} t_R(S,D,v)dDdSdv,
\]

\[
\Sigma_4 = \frac{1}{16\Delta v} \int_{v_m}^{v_M} \int_{S \in Q-R} \int_{D \in Q-R} t_R(S,D,v)dDdSdv
\]

(note that, due to symmetry reasons, \( \Sigma_2 = \Sigma_3 \)).\(^4\) In summary, we have that

\[^3\]The probability density function of a random variable is a function which describes the density of probability at each point in the sample space: it is well-known that this function is the derivative of the cdf.

\[^4\]For \( i = 1,2,3,4 \), \( \Sigma_i \) is indeed a function \( \Sigma_i(p,q) \): however, for the sake of brevity, we will always avoid to explicitly state the dependency on \( p \) and \( q \).
\[ E[T_{p,q}] = \Sigma_1 + 2\Sigma_2 + \Sigma_4. \]  

The goal of the Section 3 will be to compute the three integrals \( \Sigma_1, \Sigma_2, \) and \( \Sigma_4 \).

In order to compute \( E[T] \) we proceed by first proving a simple preliminary result computing the time spent by a node traveling along a segment with endpoints \( A \) and \( B \) according to the AC/DC motion rule. To this aim, denote as \( M \) the midpoint of \( AB \), denote as \( v \) the maximum velocity to be reached during the AC/DC motion, denote as \( a \) the acceleration, and denote as \( t_2 \) the time spent by a node to cover a segment \( \ell \). Observe now that \( t_{AM} \) is equal to the time necessary to reach the maximum velocity \( v \). Since

\[ |AM| = \frac{1}{2}t_{AM}, \]

we have that

\[ t_{AM} = \frac{|AB|}{v}. \]  

Clearly, \( t_{AB} = 2t_{AM} \); hence,

\[ t_{AB} = 2\frac{|AB|}{v}. \]  

Observe now that

\[ E[T] = \frac{1}{\Delta v} \int_{v_m}^{v} \int_{S \in Q} \int_{D \in Q} \frac{t_v(S,D)}{16} dDdSdv \]

where \( t_v(S,D) \) denotes the time spent by a node traveling along one of the two Manhattan paths from \( S \) to \( D \) according to the AC/DC motion rule. On the ground of Equation (4), we have that

\[ t_v(S,D) = 2\left( |x_S - x_D| + |y_S - y_D| \right). \]

Hence, by setting \( c = \Delta v / \ln(v_M/v_m) \), we have

\[ E[T] = \frac{1}{16c} \int_{S \in Q} \int_{D \in Q} 2\left( |x_S - x_D| + |y_S - y_D| \right) dDdS. \]

The above integral is equal to 32 times the expected Manhattan distance between two points randomly chosen within the square \( Q \); it is known that this latter value is equal to \( 2\sqrt{3} \) (see [10]). In conclusion, in the case of the AC/DC motion, we have that

\[ E[T] = \frac{8}{3c}. \]

\( \text{(5)} \)

### 3 The spatial distribution

Due to the symmetry of the AC/DC-\( x \)-MRWP model, in this section we can limit ourselves to compute \( E[T_{p,q}] \) for \( p, q < 1 \). Indeed, once we have computed the cdf and, hence, the pdf \( f_{p,q < 1}(x,y) \) relative to this case, we have that the value of \( f(x,y) \) is equal to

\[
\begin{cases}
    f_{p,q < 1}(x,y) & \text{if } x \in [0,1] \cap y \in [0,1], \\
    f_{p,q < 1}(2-x,y) & \text{if } x \in [1,2] \cap y \in [0,1], \\
    f_{p,q < 1}(x,2-y) & \text{if } x \in [0,1] \cap y \in [1,2], \\
    f_{p,q < 1}(2-x,2-y) & \text{if } x \in [1,2] \cap y \in [1,2].
\end{cases}
\]

\( \text{(6)} \)

In order to compute \( f_{p,q < 1}(x,y) \), we have to compute, for \( i = 1, 2, 4 \), the values \( \Sigma_i \) defined in the previous section. To compute \( t_{R}(S,D,v) \), for each pair of points \( S \) and \( D \), we find the intersection between \( R \) and the chosen Manhattan path from \( S \) to \( D \), and we then take the same approach used in the previous section for deriving Equations (3) and (4). To this aim, let \( A, B, M \) and \( v \) be defined as in the previous section, let \( a \) denote the acceleration, and let \( t_I \) denote the time spent by a node to cover a segment \( \ell \). If \( I \) is a generic point in \( AB \), then the following three cases arise.

- **I coincides with M or I coincides with B.** In this case, \( t_{AI} \) is computed by means of Equation (3) or (4).
- **I lies between A and M.** Since
  \[
  \begin{align*}
  |AI| &= \frac{1}{2}a^2t_{AI} \\
  a &= \frac{v}{t_{AM}} = \frac{v^2}{|AM|},
  \end{align*}
  \]
  then
  \[ t_{AI} = \sqrt{\frac{2|AI|}{a}} = \frac{1}{v} \sqrt{2|AI| \cdot |AB|}. \]

- **I lies between M and B.** Since \( t_{AI} = t_{AB} - t_{IB} \), then it is sufficient to compute \( t_{IB} \). Similarly to the previous case, we can show that
  \[ t_{IB} = \frac{1}{v} \sqrt{2|IB| \cdot |AB|}. \]

An easy exploitation of Equations (3), (4), (7) and (8) allows us to compute the value of \( t_{R} \).

**Computing \( \Sigma_1 \)** In this case, the entire Manhattan path from \( S \) to \( D \) is contained in \( R \). Hence, if \( H \) denotes the turning point of the path, we can apply Equation (4) obtaining that

\[
  t_{R}(S,D,v) = 2\left( |SH| + |HD| \right)/v.
\]

The value of \( \Sigma_1 \) is equal to

\[
  \frac{1}{16c} \int_{S \in R} \int_{D \in R} 2\left( |SH| + |HD| \right) dDdS.
\]

After having evaluated the above integral, we have that

\[
  \Sigma_1 = \frac{1}{16c} \cdot \frac{2}{3} (p^3q^2 + p^2q^3).
\]

**Computing \( \Sigma_2 \)** In order to deal with this case, we decompose \( Q - R \) into twelve regions as shown in the following figure, where \( R \) is shown in grey.
\[
\begin{array}{c|cc|cc}
D \in & I & J & \text{Implicit} & \text{Explicit} \\
\hline
R_1 & (x_D, q) & (x_S, q) & |SH| + 2|HD| - \sqrt{2|ID||HD|} & (x_S - x_D) + 2(y_D - y_S) - \sqrt{2(y_D - q)(y_D - y_S)} \\
R_2 & (x_D, q) & (x_S, q) & |SH| + 2|HT||HD| & (x_S - x_D) + \sqrt{2(q - y_S)(y_D - y_S)} \\
R_9 & (p, y_S) & (x_S, q) & |SH| + |SK| - \sqrt{2|HD| |SH| + \sqrt{2|JK||SK|}} & (x_D - x_S) + (y_D - y_S) - \sqrt{2(x_D - p)(x_D - x_S) + 2(y_D - q)(y_D - y_S)} \\
R_{10} & (p, y_S) & (x_S, q) & \frac{|SH| - \sqrt{2|SH||SK| - \sqrt{2|SK||JK|}}}{2} & (x_D - x_S) - \sqrt{2(x_D - p)(x_D - x_S) + 2(q - y_S)(y_D - y_S)} \\
R_{12} & (p, y_S) & (x_S, q) & \frac{\sqrt{2|SH||SH| + \sqrt{2|SK||JK|}}}{2} & \frac{\sqrt{2(p - x_S)(x_D - x_S) + 2(q - y_S)(y_D - y_S)}}{2} \\
\end{array}
\]

Table 1: computing \( \Sigma_2 \) (recall that \( H = (x_D, y_S) \) and \( K = (x_S, y_D) \)).

Let us consider, as an example, the case in which \( D \) is contained in \( R_1 \): in this case, we can apply Equations (4) and (8). If \( H \) and \( K \) denote the turning point of \( m_{ih}(S, D) \) and \( m_{eh}(S, D) \) respectively, and if \( I \) and \( J \) are the intersection of the line \( y = q \) with \( m_{ih}(S, D) \) and \( m_{eh}(S, D) \) respectively, we have that the value of \( t_R(S, D, v) \) is equal to

\[
\frac{1}{16c} \int_{S \in R} \int_{D \in R_1} (|SH| + 2|HD| - \sqrt{2|ID||HD|}) \, dDdS.
\]

Observe now that, since \( D \in R_1 \), then

\[
|SH| = x_S - x_D, |HD| = y_D - y_S, \quad \text{and} \quad |ID| = y_D - q.
\]

After having substituted these expressions into the integrand function and after having evaluated the above integral, we have that the contribution to \( \Sigma_2 \) given by this case is equal to

\[
\sigma_{21}(p, q) = \frac{1}{16c} \frac{p^2 q^2}{48} \left( 4p + \sqrt{2q} \ln \left( 2\sqrt{2} + 3 \right) + 12q \right).
\]

We can deal with the other eleven regions in a similar way (see Table 1 where both the implicit and the explicit integrand functions are indicated). Observe that only the contributions corresponding to \( R_1 \), \( R_2 \), \( R_9 \), \( R_{10} \), and \( R_{12} \) have to be explicitly computed, since, due to symmetry, we have that \( \sigma_{21}(p, q) = \sigma_{21}(q, p) \), \( \sigma_{22}(p, q) = \sigma_{22}(q, p) \), \( \sigma_{23}(p, q) = \sigma_{23}(q, p) \), \( \sigma_{24}(p, q) = \sigma_{24}(q, p) \), and \( \sigma_{25}(p, q) = \sigma_{25}(q, p) \), and \( \sigma_{26}(p, q) = \sigma_{26}(q, p) \), \( \sigma_{27}(p, q) = \sigma_{27}(q, p) \), \( \sigma_{28}(p, q) = \sigma_{28}(q, p) \), and \( \sigma_{29}(p, q) = \sigma_{29}(q, p) \), and \( \sigma_{210}(p, q) = \sigma_{211}(q, p) \).

After having evaluated all these contributions, we obtain that

\[
\Sigma_2 = \frac{1}{16c} \left( \frac{-(3p + 2)(p - 6)(q + 2)q\sqrt{p}}{36} - \frac{(3q + 2)(-q - 6)(p + 2)p\sqrt{q}}{36} + \frac{pq\sqrt{(10 - 19q)} + q^2(10 - 19p)}{12\sqrt{2}} \right. \\
\left. + \frac{(p - 2)(q + 2)p\ln \left( \sqrt{2 + \sqrt{p}} \right)}{24\sqrt{2}} + \frac{(q - 2)(q + 2)p\ln \left( \sqrt{2 + \sqrt{q}} \right)}{24\sqrt{2}} \right).
\]

Computing \( \Sigma_4 \) Observe that only two situations giving raise to a non empty intersection with \( R \) may occur: either \( D \) is in the region above \( R \) and \( S \) is in the region at its right or vice versa. Since the two cases are symmetric, we can limit ourselves to analyze only the first one, that is, the case in which only the path \( m_{ih}(S, D) \) intersects \( R \). To this aim, we distinguish the case in which \( p < x_S < 2p \) from the case in which \( 2p < x_S < 2 \). In the first case, we decompose the region including \( D \) into four regions as shown in the following figure.
By reasoning similarly to the previous subsection, we have that the contribution to \( \Sigma_4 \) given by this case is equal to

\[
\frac{1}{16c} \frac{1}{48} p^2 q^2 (p + q) \left( 6 + \sqrt{2} \ln \left( 1 + \sqrt{2} \right) \right).
\]

We can deal with the other three regions \( R_4, R_5, \) and \( R_6 \) in a similar way (see the first four rows of Table 2 where both the implicit and the explicit integrand functions are indicated).

In the case in which \( 2p < x_D < 2 \), we decompose the region including \( D \) into two regions as shown in the following figure.

The analysis of these two cases can be done similarly to the previous ones: their contributions to \( \Sigma_4 \) are obtained by looking at the last two rows of Table 2. After having evaluated all these contributions, we obtain that

\[
\Sigma_4 = \frac{1}{16c} 2 \left( \frac{3p + 2)(p - 6)(q - 2)q \sqrt{p}}{36} + \frac{3q + 2)(q - 6)(p - 2)p \sqrt{q}}{36} + \frac{7pq (p^2 + q^2)(p - 2)}{36} - \frac{pq (p^2 + q^2)(p - 2)}{12\sqrt{2}} \right)
\]

\[
\left( \frac{24\sqrt{2}}{(p - 2)^3 q - 2)q \ln \left( \frac{\sqrt{2} + \sqrt{p}}{\sqrt{2} - \sqrt{p}} \right)}{24\sqrt{2}} - \frac{(q - 2)^3 (p - 2)p}{24\sqrt{2}} \ln \left( \frac{\sqrt{2} + \sqrt{q}}{\sqrt{2} - \sqrt{q}} \right) \right). \tag{11}
\]

By applying Equations (1), (2) and (5) and (9)-(11), we can derive the total cdf. The pdf can be obtained by computing the derivative of the cdf: we have that, for \( 0 < x, y < 1 \),

\[
f_{p,q < 1}(x,y) = \frac{10\sqrt{2} - 3x^{3/2} - x^2}{64} + 10\sqrt{9} - 3y^{3/2} - y^2 \left( \frac{2}{64} - \frac{3\sqrt{2}(x - 2)}{256} \right) \ln \left( \frac{\sqrt{2} + \sqrt{2}}{\sqrt{2} - \sqrt{2}} \right) \ln \left( \frac{\sqrt{2} + \sqrt{2}}{\sqrt{2} - \sqrt{2}} \right) \tag{12}
\]

The right part of Figure 1 shows the behavior of \( f(x,y) \), obtained by combining the above equation and Equation (6).

### 4 Connectivity in the AC/DC MRWP model

The computation of an upper bound on the communication range proceeds similarly to [8].

In the following, \( N = \{1, \ldots, n\} \) will denote the set of \( n \) nodes moving in the square \( Q \) according to the AC/DC-\( r \)-MRWP mobility model and transmitting with range \( r \), and \( G_r(N, t) \) will denote the communication graph induced by \( N \) at time \( t \), in which two nodes are adjacent if and only if they are within the transmission range of each other, that is, within distance \( r \).

**Theorem 1.** There exist three positive constants \( \alpha, \beta, \) and \( \gamma \) such that, for any \( r = r(n) \geq \gamma \frac{\ln n}{n} \) and for any \( t \) large enough,

\[
Pr \left( G_r(N, t) \text{ is not connected} \right) < \frac{\beta}{n^\alpha}.
\]

**Proof.** In order to prove the theorem, we will follow the same approach which has been used in the case of (static) geometric random graphs (see, for example, [11]). Let \( r = \gamma \frac{\ln n}{n} \) be the transmission range of the nodes. By tessellating the square \( Q \) into \( k^2 \) square cells of size \( z = 2/k \), with \( k = \sqrt{\gamma} / r(n) \), it is not difficult to prove that, in order to guarantee the connectivity of the communication graph, it suffices to choose \( \gamma \) so that, with high probability,
every cell is not empty. For any cell \( C \) of \( \mathcal{Q} \), let \( X_i(C, t) \) be the random variable whose value is 1 if, at time \( t \), node \( i \) is in \( C \) and 0 otherwise. Moreover, let \( X(C, t) = X_1(C, t) + \ldots + X_n(C, t) \) be the random variable describing the total number of nodes in \( C \). By observing the right part of Figure 1, it follows that only the cells at the corners of \( \mathcal{Q} \) have to be analyzed. Let \( C_0 \) be the cell at the bottom left corner of \( \mathcal{Q} \) (the other corner cells can be analyzed similarly). Then

\[
\Pr[X_i(C_0, t) = 1] = \int_{Y \in C_0} f(Y) dY.
\]

This value is equal to the value \( g(z) \) of the cdf computed with input \((z, z)\), that is,

\[
g(z) = \frac{1}{384} z \left( (6 \sqrt{2} \ln(1 + \sqrt{2}) - 4) z^3 - 4(-6 + z)(2 + 3z) \sqrt{z} + 3 \sqrt{2}(-2 + z)^3 \ln \left( \frac{\sqrt{2} + \sqrt{z}}{\sqrt{2} - \sqrt{z}} \right) \right).
\]

We now prove that, for any \( z \in [0, 1] \),

\[
g(z) \geq \frac{1}{6} z^{5/2}.
\]

For the sake of simplicity, we study the following equivalent problem:

\[
g(z^2) - \frac{1}{6} z^5 \geq 0. \tag{13}
\]

After some simplifications, we obtain that

\[
g(z^2) = \frac{1}{192} z^8 \left( -2 + 3 \sqrt{2} \ln(1 + \sqrt{2}) \right) + \frac{1}{192} z^3 \left( -6 z^4 + 32 z^2 + 24 \right) + \frac{\sqrt{2}}{128} (-2 + z^3) \ln \left( \frac{\sqrt{2} + z}{\sqrt{2} - z} \right).
\]

The logarithm in the above expression can be developed into series as follows:

\[
\ln \left( \frac{\sqrt{2} + z}{\sqrt{2} - z} \right) = \sum_{k \geq 0} a_k z^{2k+1},
\]

where \( a_k = \frac{1}{2^{2k+1}} \). Therefore, if we consider the operator \([z^n] \) which acts on a formal power series \( h(z) = \sum_{k \geq 0} h_k z^k \) by extracting the coefficient of \( z^n \), that is \([z^n] h(z) = h_n \) (see, for example, [15, 16]), we have that, for \( n = 2k + 1 \geq 6 \):

\[
b_k = [z^n] (-2 + z^2)^3 \sum_{k \geq 0} a_k z^{2k+1} = [z^n] (-8 + 12 z^2 - 6 z^4 + z^6) \sum_{k \geq 0} a_k z^{2k+1} = -8 z^n \sum_{k \geq 0} a_k z^{2k+1} + 12 [z^{n-2}] \sum_{k \geq 0} a_k z^{2k+1} - 6 [z^{n-4}] \sum_{k \geq 0} a_k z^{2k+1} + [z^{n-6}] \sum_{k \geq 0} a_k z^{2k+1} = -8 a_k + 12 a_{k-1} - 6 a_{k-2} + a_{k-3}.
\]

After some simplifications, we have that, for \( k \geq 3 \),

\[
b_k = \frac{384 \sqrt{2}}{(2k - 5)(2k - 3)(2k - 1)(2k + 1) 2^k}. \tag{14}
\]

Consequently, we have that

\[
(-2 + z^2)^3 \sum_{k \geq 0} b_k z^{2k+1} = \sum_{k \geq 3} b_k z^{2k+1} - 8 \sqrt{2} + \frac{32}{3} \sqrt{2} z^3 - \frac{22}{5} \sqrt{2} z^5.
\]

The function \( g(z^2) \) can be hence rewritten as follows:

\[
g(z^2) = \left( \frac{\sqrt{2}}{64} \ln(1 + \sqrt{2}) - \frac{1}{96} \right) z^8 - \frac{z^7}{10} + \frac{z^5}{3} + \frac{\sqrt{2}}{128} z^2 \sum_{k \geq 3} b_k z^{2k+1}. \tag{15}
\]

Now, it can be easily proved that the coefficients \( b_k \) defined in Equation (14) are positive for \( k \geq 3 \). Hence in Equation (15) the series \( \sum_{k \geq 3} b_k z^{2k+1} \) gives a positive contribution when computed in \( z \in [0, 1] \). Therefore, in order to prove Equation (13) it is sufficient to prove the following inequality:

\[
\left( \frac{\sqrt{2}}{64} \ln(1 + \sqrt{2}) - \frac{1}{96} \right) z^8 - \frac{z^7}{10} + \frac{z^5}{3} - \frac{1}{6} z^5 \geq 0,
\]

or, equivalently,

\[
Q(z) = 5 \left( -2 + 3 \sqrt{2} \ln(1 + \sqrt{2}) \right) z^3 - 96 z^2 + 160 \geq 0.
\]

Now, the derivative \( Q'(z) \) of \( Q(z) \) is negative at \( z = 0 \) and \( z = \frac{64}{3} (3 \sqrt{2} \ln(1 + \sqrt{2}) - 2)^{-1} \approx 7.36 \). Hence, the polynomial \( Q(z) \) is a decreasing function for \( z \in [0, 1] \). Since \( Q(0) = 160 \) and \( Q(1) \approx 72.7 \), we have that Equation (13) holds.

Let \( r(n) = \gamma \frac{\sqrt{n}}{5} \). For any cell \( C \), let us denote by \( \mu(C, t) \) the expected value of \( X(C, t) \). We have that

\[
\mu(C, t) \geq \mu(C_0, t) = \frac{n}{6} z^2.
\]

By applying the Chernoff bound, we obtain that

\[
\Pr[X(C, t) < 1] \leq \mu(C) e^{1 - \mu(C)}.
\]

Hence, the probability that \( G_r(N, t) \) is not connected is bounded by

\[
\sum_C \Pr[X(C, t) < 1] \leq k^2 \mu(C) e^{1 - \mu(C)} \leq \frac{k^2 n}{6} z^2 e^{1 - \frac{\pi}{2} z^2} = k^2 \frac{n}{6} \frac{4 \sqrt{2}}{k^2} z^2 e^{1 - \frac{4 \sqrt{2}}{k^2} z^2} = \frac{2 \sqrt{2} n}{3 \cdot 5^4} 1 - \frac{2 \sqrt{2} n}{3 \cdot 5^4} z^2 \leq 2 \frac{\sqrt{2} n}{3 \cdot 5^4} e^{1 - \frac{2 \sqrt{2} n}{3 \cdot 5^4} z^2} \leq 2 \frac{\gamma}{3} n e^{1 - \frac{2 \sqrt{2} n}{3 \cdot 5^4} z^2}.
\]
By choosing $\gamma > 2.29$, we obtain that

$$Pr(\ G_r(N,t) \text{ is not connected } ) < \frac{\beta}{n^\alpha}$$

where $\beta$ and $\alpha$ are two positive constants.

The above theorem implies that, whenever $r(n) \geq \gamma \sqrt{\frac{\ln n}{n}}$, the probability that the graph $G_r(N,t)$ is connected is greater than $1 - \frac{\beta}{n^\alpha}$, that is, the communication graph is connected with high probability.

5 Conclusion and further research

In this paper, we have analyzed the spatial node stationary distribution of a Manhattan path based variation of the RWP mobility model. We have then applied these analytical results to the computation of an upper bound on the transmission range guaranteeing, with high probability, the connectivity of the communication graph of a MANET, whose nodes move according to the new model.

The main question left open by this paper is whether the upper bound is tight. To this aim, it seems that the second moment method (see [1]), which is usually applied to the case of random geometric graphs, leads, in our case, to the computation of very difficult integrals.

Finally, it may be also interesting to design an algorithm to sample the initial simulation state from the stationary regime of our new mobility model (see [14]).

References